Further Maths Revision Paper 1

This paper consists of 5 questions covering CP1, CP2, FP1 and FM1.
(AS Further Maths: Q1, 2 and 3)

1

Solve

$$\frac{4x+1}{x+2} \leqslant \frac{5}{x-3}, \qquad x \neq -2, x \neq 3$$

$$(4x+1)(x+2)(x-3)^2 \leq 5(x-3)(x+2)^2$$

$$(4x+1)(x+2)(x-3)^2 - 5(x-3)(x+2)^2 \leq 0$$

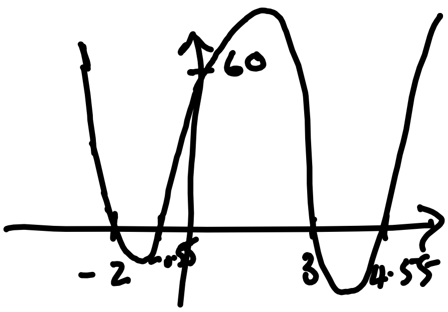
$$(x+2)(x-3)\left[(4x+1)(x-3) - 5(x+2)\right] \leq 0$$

$$(x+2)(x-3)(4x^2 - 11x - 3 - 5x - 10) \leq 0$$

$$(x+2)(x-3)(4x^2 - 16x - 13) \leq 0$$

$x = -2$, $x = 3$ $\qquad x = \frac{4 + \sqrt{29}}{2}$ $\qquad x = \frac{4 - \sqrt{29}}{2}$

$= 4.69$ $\qquad - 0.69$



$$\underline{\underline{-2 < x \leq -0.55, \quad 3 < x \leq 4.55}}$$

The tangent at a point P on the parabola $y^2 = 4ax$ meets the directrix at Q. The line through Q parallel to the x-axis meets the normal at P at the point R. Find the equation of the locus of R.

Directrix of parabola. $x = -a$.

$2y \frac{dy}{dx} = 4a$ $\qquad P(at^2, 2at)$

$\frac{dy}{dx} = \frac{4a}{2y}$

$\frac{dy}{dx} = \frac{4a}{4at} = \frac{1}{t}$

TGT

$y - 2at = \frac{1}{t}(x - at^2)$ $\qquad Q\left(-a, -\frac{a}{t} + at\right)$

$y = \frac{1}{t}(-a - at^2) + 2at$

$y = -\frac{a}{t} - at + 2at$

$y = -\frac{a}{t} + at$

$y = -\frac{a}{t} + at$

$\left(\quad , -\frac{a}{t} + at\right)$

NORMAL

$y - 2at = -t(x - at^2)$

$-\frac{a}{t} + at - 2at = -xt + at^3$

$-\frac{a}{t} - at = -xt + at^3$

$-\frac{a}{t} - at - at^3 = -xt$

$\frac{a}{t^2} + a + at^2 = x$ $\qquad \left(\frac{a}{t^2} + a + at^2, -\frac{a}{t} + at\right)$

$x = a\left(\frac{1}{t^2} + 1 + t^2\right)$ $\qquad y = a\left(t - \frac{1}{t}\right)$

$y^2 = a^2\left(t^2 - 2 + \frac{1}{t^2}\right)$

$y^2 = a^2\left(t^2 + 1 + \frac{1}{t^2}\right) - 3a^2$

$\underline{\underline{y^2 = ax - 3a^2}}$

3

Prove by induction that

$$2^{n+2} + 3^{2n+1}$$

is divisble by 7 for all positive integers.

Show true for $n = 1$

$$2^3 + 3^3 = 8 + 27 = 35 = 7(5)$$

Assume true for $n = k$

$$f(k) = 2^{k+2} + 3^{2k+1} = 7p$$

Show true for $n = k+1$

$$f(k+1) = 2^{k+3} + 3^{2k+3}$$

$$= 2(2^{k+2}) + 9(3^{2k+1})$$

$$= 2(2^{k+2} + 3^{2k+1}) + 7(3^{2k+1})$$

$$= 2(7p) + 7(3^{2k+1})$$

$$= 7(2p + 3^{2k+1})$$

Since we have shown that if it is true for $n=k$ it is true for $n=k+1$, and we have shown it is true for $n=1$, it follows by induction that it is true $\forall n \in \mathbb{Z}^+$

4

If $x = e^t$
show that
$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 16 \qquad (1)$$
reduces to
$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 4y = 16$$
Hence find the general solution for the equation (1)

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \qquad \Rightarrow \qquad x\frac{dy}{dx} = \frac{dy}{dt}$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{dt^2} \cdot \frac{dt}{dx}$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dt^2} - 4y = 16$$

$$m^2 - 4 = 0 \qquad m = \pm 2$$

P.I

$$y = Ae^{2t} + Be^{-2t}$$

C.F

$$y = p$$
$$\dot{y} = 0 \qquad \Rightarrow \quad -4p = 16$$
$$\ddot{y} = 0 \qquad\qquad p = -4$$

$$y = Ae^{2t} + Be^{-2t} - 4$$

$$y = Ax^2 + \frac{B}{x^2} - 4$$

$$x^2y = Ax^4 + B - 4x^2$$

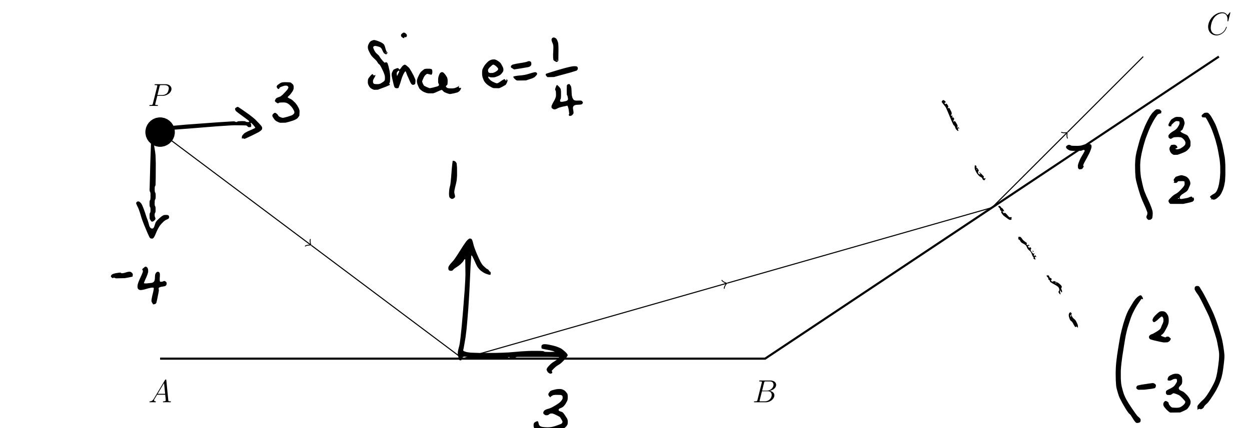


Figure 1

Figure 1 represents the plan view of a smooth horizontal floor, where AB and BC are fixed vertical walls.
The vector $\vec{AB}$ is in the direction of $\mathbf{i}$ and the vector $\vec{BC}$ is in the direction of $(3\mathbf{i} + 2\mathbf{j})$.
A small ball P is projected across the floor towards AB. immediately before the impact with AB, the velocity of P is $(3\mathbf{i} - 4\mathbf{j})\text{ms}^{-1}$.
The ball bounces off AB and then hits BC.
The ball is modelled as a particle.
The coefficient of restitution between P and AB is $\frac{1}{4}$.
The coefficient of restitution between P and BC is e.
Given that after both impacts the velocity of P is parallel to $(31\mathbf{i} + 25\mathbf{j})$ find:

(a) the value of e;

(b) the speed of P after both impacts.

Before

$$\begin{pmatrix}3\\1\end{pmatrix} = \alpha\begin{pmatrix}3\\2\end{pmatrix} + \beta\begin{pmatrix}2\\-3\end{pmatrix}$$

$$\Rightarrow \alpha = \frac{11}{13} \qquad \beta = \frac{3}{13}$$

After

$$\begin{pmatrix}31k\\25k\end{pmatrix} = \alpha\begin{pmatrix}3\\2\end{pmatrix} - e\beta\begin{pmatrix}2\\-3\end{pmatrix}$$

$$\begin{pmatrix}31k\\25k\end{pmatrix} = \frac{11}{13}\begin{pmatrix}3\\2\end{pmatrix} - \frac{3}{13}e\begin{pmatrix}2\\-3\end{pmatrix}$$

$$31k = \frac{33}{13} - \frac{6e}{13} \Rightarrow 403k = 33 - 6e$$

$$25k = \frac{22}{13} + \frac{9e}{13} \Rightarrow 325k = 22 + 9e$$

$$k = \frac{1}{13} \qquad e = \frac{1}{3}$$

After

$$\begin{pmatrix}\frac{31}{13}\\ \frac{25}{13}\end{pmatrix}$$

$\Rightarrow$ Speed

$$= \sqrt{\left(\frac{31}{13}\right)^2 + \left(\frac{25}{13}\right)^2}$$

$$= \sqrt{\frac{122}{13}} = 3.06$$