Further Maths Revision Paper 4

This paper consists of 5 questions covering CP1, CP2, FP1 and FM1.
(AS Further Maths: Q1 and 3)

1

$$P = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$$

The matrix P represents a linear transformation, T, of the plane.

(a) Describe the invariant points of the transformation T.

(b) Describe the invariant lines of the transformation T.

a) $\begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$4x - 2y = x$ $\qquad$ $3x - 2y = 0$

$3x - y = y$ $\qquad$ $3x = 2y$

$\underline{\underline{y = \frac{3}{2}x}}$

All points on the line $\underline{\underline{y = \frac{3}{2}x}}$

b) $\begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} x' \\ mx'+c \end{pmatrix}$

$4x - 2mx - 2c = x'$

$3x - mx - c = mx' + c$

$3x - mx - c = 4mx - 2m^2x - 2mc + c$

$x(3 - m - 4m + 2m^2) - c - c + 2mc = 0$

$(3 - 5m + 2m^2) = 0$ $\qquad$ $-2c + 2mc = 0$

$m = \frac{3}{2}$ $\quad$ $m = 1$

$m = 1$ $\qquad$ $-2c + 2c = 0$

$\Rightarrow$ $y = x + k$ is invariant $\forall k$

$m = \frac{3}{2}$ $\qquad$ $-2c + 3c = 0$

$c = 0$ $\Rightarrow$

$\underline{\underline{y = \frac{3}{2}x \text{ is invariant}}}$ as before

2

The point P lies on the hyperbola with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

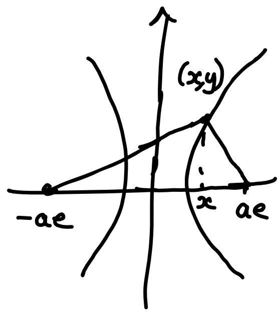
The points S and S' are the foci of the hyperbola.
Show that $S'P - SP = 2a$

foci $S(ae, 0)$ $S'(-ae, 0)$

$b^2 = a^2(e^2-1)$ for hyperbola

$b^2 = a^2e^2 - a^2$

$y^2 = \left(\frac{x^2}{a^2} - 1\right)b^2$



$y^2 = \left(\frac{x^2}{a^2} - 1\right)a^2(e^2-1)$

$= (x^2 - a^2)(e^2-1)$

$= x^2e^2 - e^2a^2 + a^2 - x^2$

$P(x, y)$

$SP = \sqrt{(ae-x)^2 + y^2}$

$= \sqrt{a^2e^2 - 2aex + x^2 + y^2}$

$= \sqrt{\cancel{a^2e^2} - 2aex + \cancel{x^2} + x^2e^2 - \cancel{e^2a^2} + a^2 - \cancel{x^2}}$

$= \sqrt{x^2e^2 - 2aex + a^2}$

$= \sqrt{(xe-a)^2} = xe - a$

$S'P = \sqrt{(ae+x)^2 + y^2}$

$= \sqrt{a^2e^2 + 2aex + x^2 + y^2}$

$= \sqrt{\cancel{a^2e^2} + 2aex + \cancel{x^2} + x^2e^2 - \cancel{e^2a^2} + a^2 - \cancel{x^2}}$

$= \sqrt{x^2e^2 + 2aex + a^2}$

$= \sqrt{(xe+a)^2} = xe + a$

$S'P - SP = xe + a - (xe - a)$

$= \underline{\underline{2a}}$

3

(a) Obtain the Cartesian equation of the straight line which passes through the point $A(-1, 2, 3)$ and which is normal to the plane $2x - 3y + 4z + 8 = 0$

(b) Calculate the coordinates of P the point of the intersection of this line with the plane.

(c) If the point $B(a, 2a, 3)$ lies on the plane, find the value of a and calculate the angle between AP and AB in degrees giving your answer to 1 decimal place.

a)

$$l_1: \ \underline{r} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

$$\frac{x+1}{2} = \frac{y-2}{-3} = \frac{z-3}{4}$$

b)

$$\begin{pmatrix} -1+2\lambda \\ 2-3\lambda \\ 3+4\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = -8$$

$$-2+4\lambda-6+9\lambda+12+16\lambda = -8+2+6-12$$

$$29\lambda = -12$$

$$\lambda = \frac{-12}{29}$$

$$-1+2\lambda = \frac{-53}{29}$$

$$2-3\lambda = \frac{94}{29}$$

$$3+4\lambda = \frac{39}{29}$$

$$\left(\frac{-53}{29}, \frac{94}{29}, \frac{39}{29}\right)$$

c)

$$\begin{pmatrix} a \\ 2a \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = -8$$

$$2a - 6a + 12 = -8$$

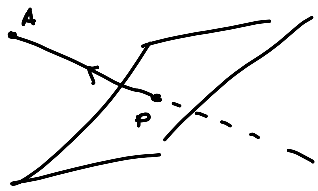
$$-4a = -20$$

$$\underline{\underline{a = 5}}$$

$B(5, 10, 3)$

$$\overrightarrow{AB} = \begin{pmatrix} 5 \\ 10 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix}$$

$$\overrightarrow{AP} = \begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix}$$



$$\begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix} = -12 + 24 = 12$$

$$|\overrightarrow{AB}| = 10 \qquad |\overrightarrow{AP}| = \sqrt{4+9+16} = \sqrt{29}$$

$$12 = 10\sqrt{29}\cos\theta$$

$$\cos\theta = \frac{12}{10\sqrt{29}} \qquad \theta = \underline{\underline{77.1^\circ}}$$

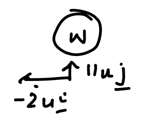
4

A red ball is stationary on a rectangular billiard table $OABC$.
It is then struck by a white ball of equal mass and equal radius with velocity $u(-2\mathbf{i} + 11\mathbf{j})$ where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along OA and OC respectively.
After impact the red and white balls have velocities parallel to the vector $-3\mathbf{i}+4\mathbf{j}$, $2\mathbf{i}+4\mathbf{j}$ respectively.

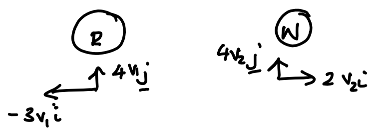
(a) Show that the lines of centres on impact is parallel to $-3\mathbf{i} + 4\mathbf{j}$

~~(b) Hence or otherwise prove that that the coefficient of restitution between the two balls is $\frac{1}{2}$~~

Before



After.



Cons mom:

$$(-2u\underline{i} + 11u\underline{j}) = -3v_1\underline{i} + 4v_1\underline{j} + 2v_2\underline{i} + 4v_2\underline{j}$$

$$-2u = -3v_1 + 2v_2 \qquad (\times 2)\ -4u = -6v_1 + 4v_2$$

$$11u = 4v_1 + 4v_2$$

$$-15u = -10v_1$$

$$v_1 = 1.5u$$

$$11u = 6u + 4v_2$$

$$5u = 4v_2$$

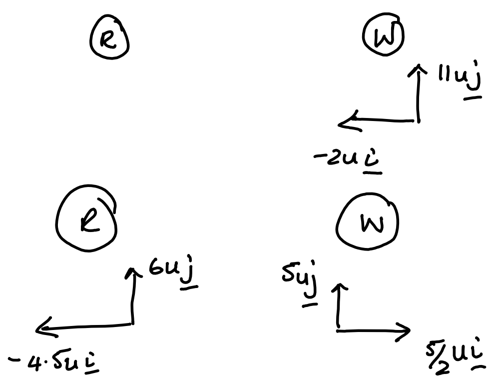
$$v_2 = \frac{5}{4}u$$

Impulse

$$m\Delta u = -4.5u\underline{i} + 6u\underline{j}$$

$\Rightarrow$ Line of centres

$$\parallel \text{ to } \begin{pmatrix} -4.5 \\ 6 \end{pmatrix}$$



$$\parallel \text{ to } \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

5

Use Taylor's theorem to evaluate

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{(x - \frac{\pi}{2})}$$

You may use:

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$

5) $\lim_{x \to \pi/2} \frac{\cos x}{(x - \pi/2)}$

$f(x) = \cos x \qquad f'''(x) = -\sin x$

$f'(x) = \sin x$

$f''(x) = -\cos x$

$\cos x = \cos \frac{\pi}{2} + \sin \frac{\pi}{2} (x - \frac{\pi}{2}) - \frac{\cos \frac{\pi}{2} (x - \frac{\pi}{2})^2}{2!} - \frac{\sin \frac{\pi}{2} (x - \frac{\pi}{2})^3}{6!}$

$= 0 - (x - \frac{\pi}{2})$

$$\frac{-(x - \frac{\pi}{2}) - \frac{1}{6}(x - \frac{\pi}{3})^3}{(x - \frac{\pi}{2})}$$

$= -1 - \frac{1}{6}(x - \frac{\pi}{2})^2 + O(x^3)$

$\Rightarrow \lim_{x \to \pi/2} \frac{\cos x}{(x - \pi/2)} = -1$